

# Entanglement of an alternating bipartition in spin chains: Relation with classical integrable models

Raul A. Santos\*

*Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot, Israel; Department of Physics, Bar-Ilan University, Ramat Gan, 52900, Israel*

We study the entanglement properties of a class of ground states defined by matrix product states, which are generalizations of the valence bond solid (VBS) state in one dimension. It is shown that the transfer matrix of these states can be related to representations of the Temperley-Lieb algebra, allowing an exact computation of Renyi entropy. For an alternating bipartition, we find that the Renyi entropy can be mapped to an eight vertex model partition function on a rotated lattice. We also show that for the VBS state, the Renyi entropy of the alternating partition is described by a critical field theory with central charge  $c = 1$ . The generalization to  $SU(n)$  VBS and its connection with a dimerization transition in the entanglement Hamiltonian is discussed.

In recent years it has become clear that any attempt for classification of quantum systems should account for their entanglement properties [1–4]. This approach has been specially fruitful in gapped systems where the energy gap induces some robustness against local perturbations. For temperatures well below the gap, the system resides in its ground state, making this state specially important in the description of a quantum system. An efficient way to describe entangled ground states is by Tensor Product States (TPS). They provide a useful representation of short [5] and long range entangled states [6–8]. In one dimension, an specific type of TPS called matrix product states (MPS) provides a way to construct any ground state of a gapped system [9, 10].

In this context, a characterization of the ground state entanglement becomes very necessary. Among the different measures of entanglement in quantum systems, a particularly useful one is the entanglement entropy (EE) (or its generalization as a Renyi entropy). In pure states, like the ground state, EE gives a unique measure of the entanglement present between two complementary subsystems.

In this letter we study a class of one dimensional spin ground states that generalizes the VBS states. These states, defined by their matrix product representation, are ground states of parent Hamiltonians which are non integrable. Despite of their lack of integrability the Renyi entropy of a partition sensitive to the bulk entanglement, introduced in Ref. [11], can be mapped to partition functions of integrable models.

This paper is organized as follows. First we review the construction of the (VBS) state in one dimension and its MPS, particularly in the context of the AKLT [12] model. We also review the computation of correlation functions and entanglement measures (Renyi and entanglement entropy) in the ground state by means of the transfer matrix technique. In the second section we propose an MPS that contains the VBS state as a particular limit. For these generalized states, we show that

a particular bipartition, dubbed alternating bipartition (AB), maps naturally to the partition function of an eight vertex model. This classical two dimensional model has analytic solution as its Boltzmann weights satisfy the Yang-Baxter equation [13]. We find that for the AKLT ground state, the corresponding eight vertex model becomes critical. The connection is made by showing that the transfer matrix of the VBS state is related with a representation of the Temperley-Lieb algebra [14]. This allows further characterization of the AB entanglement as the free energy of a conformal field theory (CFT) with central charge  $c = 1$ . Considering the generalization of the VBS state for larger symmetry groups  $SU(n)$ , we find that the Renyi entropy in this case is described by the partition function of the  $n^2$ -state Potts model, at the transition point. This transition is first order for  $n > 2$ , so is not described by a CFT. Finally, we discuss the existence of a representative among this generalized MPS whose AB entanglement is given by a topological quantity. It corresponds to a sum over loop configurations on the induced two dimensional lattice, which depends on the boundary conditions of the spin chain.

## I. AKLT MODEL AND MATRIX PRODUCT REPRESENTATION

The AKLT model is considered to be the first example of a matrix product state. It was introduced in [12] in order to study the Haldane conjecture [15] for integral spin chains. The model consists of a spin chain where at each site of the lattice resides a spin 1 particle. The Hamiltonian for this model is the sum of projection operators of nearest neighbor spins onto total spin 2. Specifically, for periodic boundary conditions, the Hamiltonian of an  $N$ -sites chain reads

$$H = \frac{1}{2} \sum_{i=1}^N \left( \frac{2}{3} + S_i \cdot S_{i+1} + \frac{1}{3} (S_i \cdot S_{i+1})^2 \right). \quad (1)$$

The ground state of this Hamiltonian is easily found by the VBS construction. At each site, spin 1 states are

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\* raul.santosza@biu.ac.il

constructed by projecting the tensor product of two spin  $\frac{1}{2}$  representations onto the symmetric subspace. This reproduces the spin 1 Hilbert space at each site. The spin  $\frac{1}{2}$  representations at adjacent sites are antisymmetrized forming singlets. This prevents the formation of spin 2 states between nearest neighbors. Hence the total spin between adjacent sites cannot be larger than 1 ( $|\vec{S}_i + \vec{S}_{i+1}| = 0, 1$ ), thus being annihilated by the AKLT Hamiltonian (1). As this Hamiltonian is a sum of projectors, its eigenvalues are semi positive definite, with the ground state having eigenvalue zero. This shows that the VBS state is the ground state of (1).

The matrix product representation of the VBS state has been studied in many contexts (see for example [16, 17]). For completeness here we rederive it. It is simply found from the previous discussion as follows. We define the matrices  $A^{[-1]}, A^{[0]}$  and  $A^{[1]}$ , related with the symmetric subspace of the tensor product of two spin  $\frac{1}{2}$  states by (sum over repeated indices implied, otherwise noted)

$$\left| \frac{1}{2}, a \right\rangle_i \otimes \left| \frac{1}{2}, b \right\rangle_i + \left| \frac{1}{2}, b \right\rangle_i \otimes \left| \frac{1}{2}, a \right\rangle_i \equiv A_{ab}^{[m]} |1, m\rangle_i, \quad (2)$$

where  $|1, m\rangle$  is a state of definite spin 1 and  $m = S_z$ . Here each label  $a$  and  $b$  takes two values  $\pm\frac{1}{2}$ . Antisymmetrization of neighboring sites is achieved by inserting the totally antisymmetric tensor  $\epsilon_{ab}$  between them

$$A_{ab}^{[m]} |1, m\rangle_i \epsilon_{bc} A_{cd}^{[m']} |1, m'\rangle_{i+1}. \quad (3)$$

Defining finally  $A^{[m]} \epsilon = M^{[m]}$ , we find for the VBS state  $|\Omega\rangle$  its matrix product representation

$$|\Omega\rangle = \text{Tr}(M^{[m_1]} M^{[m_2]} \dots M^{[m_N]}) |m_1, m_2, \dots, m_N\rangle, \quad (4)$$

where we have used the notation  $|1, m\rangle \rightarrow |m\rangle$ . The state  $|m_1, \dots, m_N\rangle$  is the tensor product over all lattice sites. The matrices  $M^{[k]}$ , written in terms of Pauli matrices  $\sigma^\pm = \sigma^x \pm i\sigma^y$ ,  $\sigma^z$  are explicitly (up to an overall factor)

$$M^{[-1]} = -\sqrt{2}\sigma^-, \quad M^{[1]} = \sqrt{2}\sigma^+, \quad M^{[0]} = -\sigma^z.$$

The matrices  $M^{[k]}$  define a map from an auxiliary space to the physical Hilbert space of spin states.

A convenient way to think of MPS is to represent them diagrammatically. We start by representing the matrices  $M$  as three legged objects (see Fig (1i)). The two horizontal lines are called bonds and the state  $|\Omega\rangle$  is obtained by contracting  $N$  of this tensors along the horizontal bonds (see Fig (1iii)). The computation of correlation functions becomes (assuming  $j > i$ )

$$\frac{\langle \Omega | S_i^a S_j^b | \Omega \rangle}{\langle \Omega | \Omega \rangle} = \frac{\text{Tr}(E^{N-j+i-1} B_i(a) E^{j-(i+1)} B_j(b))}{\text{Tr} E^N}, \quad (5)$$

where  $E = \sum_k \bar{M}^{[k]} \otimes M^{[k]}$  and  $B_i(a) = \sum_{k_i, m_i} \bar{M}^{[k_i]} \otimes M^{[m_i]} |k_i| S_i^a |m_i\rangle$ . Pictorially  $E$  is represented by Fig. (1ii). Due to the form of (5) we call  $E$  the one dimensional transfer matrix. The correlation function is represented in Fig (1iv).

The MPS construction can describe *any* one dimensional state [9, 10] for sufficiently large matrices (whose size is called bond dimension).

In the VBS case, the one dimensional transfer matrix is  $E_{(a,c);(b,d)} = \bar{M}_{ab}^{[k]} M_{cd}^{[k]}$  with

$$E_{\text{VBS}} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}. \quad (6)$$

In the thermodynamic limit of infinite sites  $\langle S_i^a S_j^b \rangle = \frac{4}{3} \left(-\frac{1}{3}\right)^{|i-j|} \delta_{ab}$ . In [18], the authors proved that (1) possess a gap in the spectrum in the thermodynamic limit. This is in agreement with the exponential decay of correlation functions with distance.

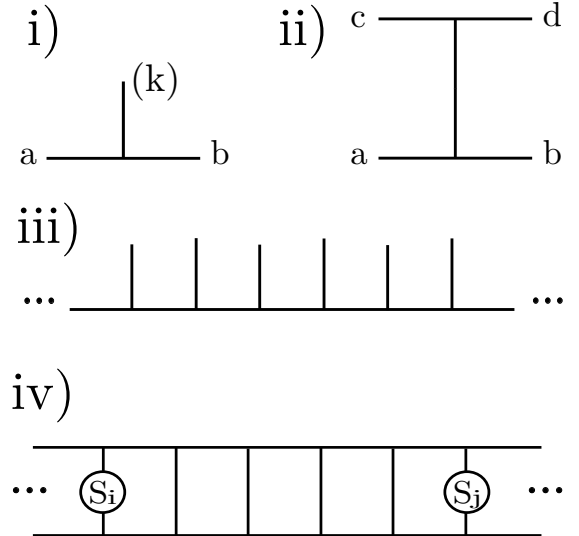


FIG. 1. Pictorial representation of i) Matrix  $M^{[k]}$  ii) One dimensional transfer matrix, iii) MPS Ground state, iv) Spin-spin correlation function.

## A. Entanglement and Renyi Entropies

The previous constructions based on MPS are also useful to investigate the entanglement properties of ground states. In the VBS case, this has been studied by many authors and in various contexts [19–22]. Here we review some of these earlier results.

Entanglement between two subsystems of a pure state is characterized by its entanglement entropy (EE), that corresponds to the von-Neumann entropy of one of the

subsystems [23]. It is defined by  $S_{\text{ent}} = -\text{Tr}(\rho_A \ln \rho_A)$ , where  $A$  is one of the subsystems. The partial density matrix is as usual  $\rho_A = \text{Tr}_{A_c} |\Omega\rangle\langle\Omega|$  and  $A_c$  is the complement of  $A$ . Note that for pure states  $S_{\text{ent}}(A) = S_{\text{ent}}(A_c)$ . A generalization of EE that allows to obtain the eigenvalues of the partial density matrix is the Renyi or  $\alpha$ -entropy, defined as

$$S_\alpha = \frac{\ln \text{Tr} \rho_A^\alpha}{1 - \alpha}, \quad (7)$$

in the limit  $\alpha \rightarrow 1$ , Renyi entropy becomes the entanglement entropy.

Defining a partition of the spin chain into two blocks of length  $k$  (subsystem  $A$ ) and  $N - k$  ( $A_c$ ), the partial density matrix is in general (using periodic boundary conditions)

$$\rho_A = \bigotimes_{i,j=1}^k \text{Tr}(E^{N-k} M^{[m_i]} \bar{M}^{[n_j]} |m_i\rangle\langle n_j|), \quad (8)$$

where we have used the MPS representation for the ground state. Note that (8) contains  $k$  matrices  $M$  and  $\bar{M}$ . Pictorially  $\rho_A$  is shown in Fig. (2i). Contracting  $\alpha$  of these objects in the vertical direction (i.e contracting the states in the physical space) we obtain  $\rho_A^\alpha$  (see also Fig (2ii)).

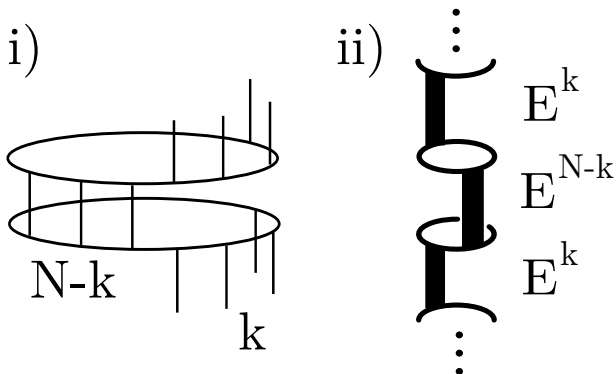


FIG. 2. i) Partial density matrix for periodic boundary conditions. Here  $k = 4$  and  $N = 7$ . ii) Contracting vertically  $\rho_A$   $\alpha$ -times we obtain  $\rho_A^\alpha$ . Thick black lines correspond to powers of transfer matrix  $E$ .

The Renyi entropy is found to be [22, 24] (in the thermodynamic limit)

$$S_\alpha(k) = \frac{\ln[(1 + 3(-\frac{1}{3})^k)^\alpha + 3(1 - (-\frac{1}{3})^k)^\alpha] - 2\alpha \ln 2}{1 - \alpha}.$$

The EE in the double scaling limit ( $k, N - k \rightarrow \infty$ ) is  $S_{\text{ent}} = 2 \ln 2$ . Here we find that EE saturates to a constant which is a consequence of the area law for EE of gapped systems in one dimension [25, 26].

## II. ALTERNATING BIPARTITION AND MAP TO EIGHT VERTEX MODEL

In the previous section we reviewed some results about the entanglement of a bipartition defined by two consecutive blocks of length  $k$  and  $N - k$ . In this section we proceed to investigate the entanglement of a different bipartition defined as the partition between odd and even sites, that we call alternating bipartition (AB). More specifically, let's number the sites in the chain from 0 to  $N - 1$  for a chain of length  $N$  and assume periodic boundary conditions. Sites at even positions belong to subset  $A$  and sites at odd positions belong to  $A_c$  (See Fig. (3)). We assume the total number of sites  $N$  to be even.

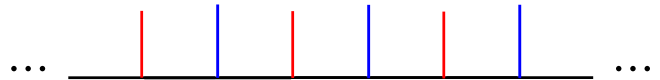


FIG. 3. (Color online) Alternating bipartition. Sites in red belong to set  $A$  while sites in blue belong to the complement set  $A_c$ .

The partial density matrix  $\rho_A$  is in this case given by Fig (4). To compute the Renyi entropy we have to stack  $\alpha$  of these objects in the vertical direction. Taking the trace generates a two dimensional grid that can be viewed as a two dimensional classical model.

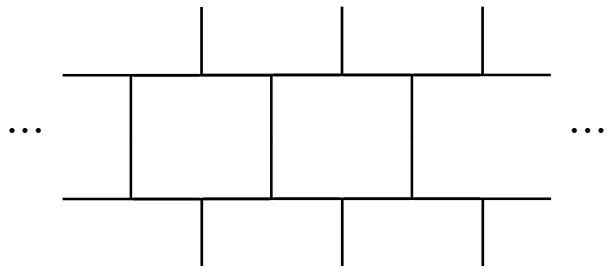


FIG. 4. Partial density matrix for the alternating bipartition. This operator can be regarded as the transfer matrix of a two dimensional classical model.

In order to study the effective classical model defined by  $\rho^\alpha$  can consider the more general transfer matrix ( $u, v, w, t \in \mathbb{R}$ )

$$E = \bar{M}^{[k]} \otimes M^{[k]} = \begin{pmatrix} u & 0 & 0 & w \\ 0 & v & t & 0 \\ 0 & t & v & 0 \\ w & 0 & 0 & u \end{pmatrix}, \quad (9)$$

which corresponds in general to the MPS of a spin  $\frac{1}{2}$  ladder with groundstate  $|\psi_0\rangle = \text{tr}(g_1 g_2 \dots g_{N-1} g_N)$ , where

$$g_i = \sum_k M^{[k]} |k\rangle = M^{[s]} |s\rangle + M^{[0]} |0\rangle + M^{[+]} |+\rangle + M^{[-]} |-\rangle. \quad (10)$$

Here  $|s\rangle$  corresponds to the singlet state along the rung, while  $|-\rangle, |0\rangle, |+\rangle$  corresponds to one of the triplet states with  $S_z = m = -1, 0, 1$  respectively. The total  $S_z$  measures the spin in the rung of the ladder. The explicit form of the  $M^{[k]}$  matrices is

$$M^{[s]} = \sqrt{\frac{u+v}{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M^{[-]} = \sqrt{w} \begin{pmatrix} 0 & \sin \theta \\ -\cos \theta & 0 \end{pmatrix},$$

$$M^{[0]} = \sqrt{\frac{u-v}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M^{[+]} = \sqrt{w} \begin{pmatrix} 0 & \cos \theta \\ -\sin \theta & 0 \end{pmatrix}.$$

where  $\sin 2\theta = -t/w$ . For  $u = -v = 1, t = 0$  and  $w > 0$  we recover the MPS (and consequently the transfer matrix) of the KSZ model [27, 28] ground state. Within the KSZ family, the point  $w = 2$  corresponds to the VBS state MPS.

In the case  $u > |v|$  and  $w > 0$  the MPS (10) is invariant under time reversal. The symmetry of this MPS is enlarged to  $SO(2) \sim U(1)$  (rotations in the xy plane) for  $u = -v = 1, t = 0$  and  $w > 0$ .

Mapping the transfer matrix (TM) four legged tensor to an arrow configuration (Fig. (5)), we can translate the computation of the Renyi ( $\alpha$ ) entropy to the partition function of the eight vertex model in a lattice with height  $\alpha$  and width  $N$ , for a spin chain of length  $N$ . Specifically, defining the Boltzmann weights of the eight vertex model as  $\omega_i$  for the  $i$ -th arrow configuration, we establish (using  $\pm \frac{1}{2} \rightarrow \pm$ )

$$E_{(++):(++)} \equiv \omega_1, \quad E_{(+-):(-+)} \equiv \omega_3, \quad (11)$$

$$E_{(+ -):(- +)} \equiv \omega_5, \quad E_{(++):(--)} \equiv \omega_7. \quad (12)$$

Negating all the indices of  $E$ , and using that it has the form (9), we find that  $\omega_1 = \omega_2 = u, \omega_3 = \omega_4 = v, \omega_5 = \omega_6 = t$  and  $\omega_7 = \omega_8 = w$ . These conditions correspond to the eight vertex model in zero field [13].

After taking the trace in (7), the Renyi entropy becomes equivalent (up to an overall constant) to the free energy of the eight vertex model in a lattice rotated  $\pi/4$  radians (45 degrees) w.r. to the square lattice. This rotation is irrelevant for a classical system with open boundary conditions, but due to the periodic b.c. of the spin chain and the trace in (7), we need to consider the classical partition function in a lattice wrapped on a torus.

The Renyi entropy is then

$$S_\alpha = \frac{\ln Z_{N,\alpha}(u, t, v, w) - N\alpha f_0(u, t, v, w)}{1 - \alpha}, \quad (13)$$

where  $N$  is the total number of sites and  $Z_{N,\alpha}(u, t, v, w)$  is the eight vertex model partition function

$$Z_{N,\alpha}(u, t, v, w) = \sum_{\mathcal{C}} u^{n_1+n_2} v^{n_3+n_4} t^{n_5+n_6} w^{n_7+n_8}, \quad (14)$$

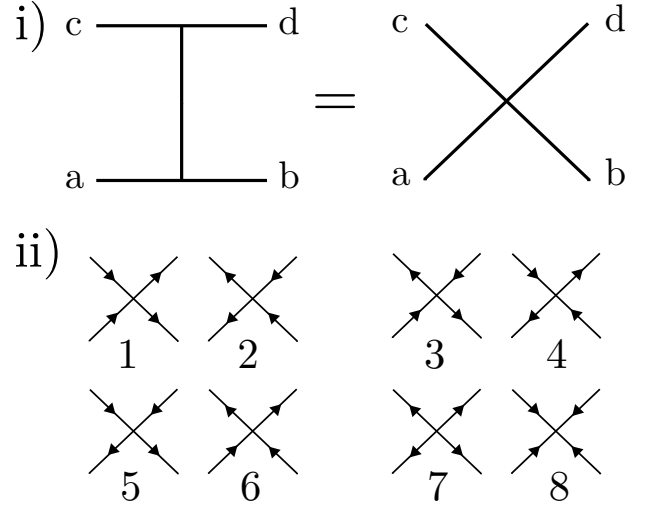


FIG. 5. i) Transfer matrix as an arrow configuration. ii) Eight possible configurations with even number of arrows of the same type. In our notation, the arrows pointing North-East and South-East correspond to the label '+', North-West and South-West are assigned the label '-'

with  $n_i$  the number of  $i$  arrow configurations. The sum runs over all the allowed configurations of arrows  $\mathcal{C}$  in the lattice. The last factor in (13),  $f_0$ , which comes from the normalization of the partial density matrix, is given by

$$f_0(u, t, v, w) = \ln \max(|u \pm v|, |w \pm t|), \quad (15)$$

where the maximum has to be taken over all four possible choices of signs.

Note the lattice where the eight vertex model is defined in (14) is rotated clockwise 45 degrees respect to the usual square lattice orientation.

#### A. Reduction to a Rotated Six Vertex Model

To make further progress we use the symmetry of the partition function (14). For  $N$  even, (which is the case we are considering), the lattice can be divided into two sublattices  $S_1$  and  $S_2$  such that every vertex in  $S_1$  has neighbors just in  $S_2$ , and viceversa. Reversing all arrows on the edges that start on a site in  $S_1$  and go south west or south east, we get a new zero field eight vertex model, but with  $u, t, v, w$  replaced by  $v, w, u, t$ , so

$$Z_{N,\alpha}(u, t, v, w) = Z_{N,\alpha}(v, w, u, t) \quad (16)$$

In the VBS case, and in its KSZ generalization, we have  $t = 0$ . Focusing in this case,  $Z_{N,\alpha}(v, w, u, 0)$  corresponds to the partition function of the six vertex model solved in [29]. This connection between the KSZ MPS and the six vertex model can be further understood comparing the symmetries of these two systems. The six vertex effective

Hamiltonian (defined as the logarithmic derivative of the transfer matrix operator) corresponds to the XXZ model which has  $U(1)$  symmetry (rotations in the xy plane), as it is also the KSZ transfer matrix (9).

### B. Alternating Entanglement of VBS as a CFT

As discussed previously, for the MPS states studied here, the Renyi entropy of the entanglement bipartition can be mapped to a classical vertex model.

In this section we will show that the VBS case is special as it maps to a critical point in the vertex model parameter space. A simple way to see this connection is the following. We introduce the operators (see also Fig. 6)

$$V_{2i} = e_0 \cdots \otimes e_{2i-1} \otimes E_{2i} \otimes e_{2i+2} \cdots \otimes e_{N-1}, \quad (17)$$

$$W_{2i+1} = e_0 \cdots \otimes e_{2i} \otimes E_{2i+1} \otimes e_{2i+3} \cdots \otimes e_{N-1}, \quad (18)$$

with  $E_i$  the one dimensional transfer matrix acting at space  $i$  and  $i+1$ . Here  $e_i$  is a  $2 \times 2$  identity matrix at site space  $i$ .

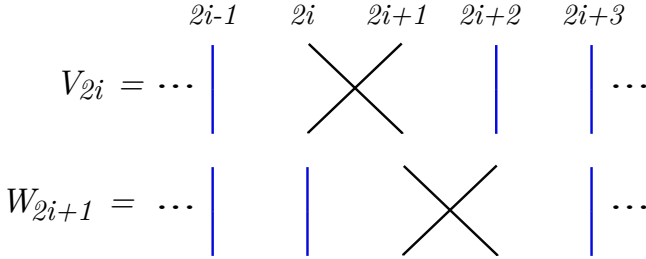


FIG. 6. Operators  $V_{2i}$  and  $W_{2i+1}$ .  $V_{2i}$  ( $W_{2i+1}$ ) acts nontrivially just in adjacent spaces  $2i$  and  $2i+1$  ( $2i+1$  and  $2i+2$ ). In these spaces  $V_{2i}$  ( $W_{2i+1}$ ) acts as the transfer matrix  $E$ . Each blue line represents the identity in the corresponding space.

The transfer matrix of the two dimensional system (see Fig 4) is then given by  $\mathcal{V}\mathcal{W}$ , where

$$\mathcal{V} = \prod_{i=0}^{(N-2)/2} V_{2i} \quad \text{and} \quad \mathcal{W} = \prod_{i=0}^{(N-2)/2} W_{2i+1}. \quad (19)$$

Plugging parametrization of the transfer matrix (see Fig. 7i)

$$E_{(ac);(bd)} = \delta_{ac}\delta_{bd} + \beta\delta_{cd}\delta_{ab}, \quad (20)$$

on the definitions of  $\mathcal{V}$  and  $\mathcal{W}$  (17), we have ( $\beta = -\frac{1}{2}$  for the VBS case)

$$\mathcal{V} = \prod_{i=0}^{\frac{N}{2}-1} (I + \beta U_{2i}), \quad \mathcal{W} = \prod_{i=0}^{\frac{N}{2}-1} (I + \beta U_{2i+1}). \quad (21)$$

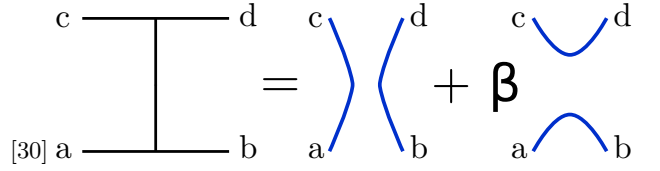


FIG. 7. i) Transfer matrix as linear combination of Temperley-Lieb generators. ii) Graphical definition of operator  $U_i$ . It acts nontrivially just in the spaces  $i$  and  $i+1$ . In the top and bottom figures, each blue line represents a Kronecker  $\delta$  function

The operators  $U_0, \dots, U_{N-1}$  are defined diagrammatically in Fig. (7ii). They satisfy the Temperley-Lieb (TL) algebra [14]

$$U_i^2 = \sqrt{q}U_i, \quad (22)$$

$$U_i U_{i+1} U_i = U_i, \quad (23)$$

$$U_i U_j = U_j U_i, \quad |i-j| \geq 2. \quad (24)$$

with  $q = 4$ . This is a particular representation of the TL algebra. As noted in [14], an equally good representation of the algebra can be achieved by the  $2^N \times 2^N$  matrices

$$(U_{2i-1}^R)_{\sigma, \sigma'} = q^{-\frac{1}{2}} \prod_{j=1 \neq i}^N \delta_{\sigma_j, \sigma'_j}, \quad (25)$$

$$(U_{2i}^R)_{\sigma, \sigma'} = q^{\frac{1}{2}} \delta_{\sigma_i, \sigma'_{i+1}} \prod_{j=1}^N \delta_{\sigma_j, \sigma'_j}. \quad (26)$$

Writing down the matrices  $\mathcal{V}$  and  $\mathcal{W}$  as before, but with this representation of the matrices  $U$ , they can be expressed as [14, 30]

$$\mathcal{V}_{\sigma, \sigma'}^R = \exp \left( K_1 \sum_{j=1}^{N-1} \delta_{\sigma_j, \sigma'_{j+1}} \right) \prod_{j=1}^N \delta_{\sigma_j, \sigma'_j}, \quad (27)$$

$$\mathcal{W}^R = \exp \left( K_2 \sum_{j=1}^N \delta_{\sigma_j, \sigma'_j} \right). \quad (28)$$

with  $K_1 = \ln(1 + \beta\sqrt{q})$  and  $K_2 = \ln(1 + \frac{1}{\beta\sqrt{q}})$ . The matrices  $\mathcal{V}^R, \mathcal{W}^R$  build up the partition function of the Potts models at the criticality. The critical manifold is defined when  $(e^{K_1} - 1)(e^{K_2} - 1) = q$  (note that this condition is independent of the parameter  $\beta$  in (20)). For the VBS case,  $q = 4$  and  $\beta = -1/2$ , so the corresponding Potts model is ferromagnetic, critical and at zero temperature. Given that the partition function does not depend on the particular representation of the TL generators [14], we find that the Renyi entropy is effectively described by a conformal field theory of central charge  $c = 1$  [30–33] which describes the Potts model at criticality for  $q = 4$  (for  $N \rightarrow \infty$  and  $\alpha \gg 1$ ).

### C. Generalization to $SU(n)$

A natural generalization of the previous discussion is to consider  $SU(n)$  invariant VBS states, where  $n = 2$  represents the original AKLT ground state. Here we review the construction of such states following [22, 34]. At each lattice site of a chain, we consider particles in the adjoint representation of  $SU(n)$  (in the  $SU(2)$  case this corresponds to spin 1 particles). By analogy with the original AKLT case, we construct such representations by considering the tensor product of the fundamental and its conjugate representation (antifundamental) of  $SU(n)$  in each site,  $\square \otimes \bar{\square} = \text{singlet} \oplus \text{adjoint}$ , and projecting this product on the adjoint irrep subspace, (similar to constructing spin 1 irrep out of two spin  $\frac{1}{2}$  in the original case). To complete the construction we form singlets between nearest neighbor pairs of fundamental and conjugate representations. Choosing a basis for the states in the fundamental representation  $|a\rangle$  ( $a = 1, \dots, n$ ) and  $|\bar{a}\rangle$  for states in the conjugate, the projector onto the adjoint representation can be written as

$$\mathcal{P}_{\text{adj}} = \sum_{a, \bar{a}} |a, \bar{a}\rangle \langle a, \bar{a}| - \frac{1}{n} \sum_{a, b} |a, \bar{a}\rangle \langle b, \bar{b}|, \quad (29)$$

where we have used that the representation of  $\square \otimes \bar{\square}$  splits into the singlet and adjoint irreps. The projector into the singlet state is just  $|0\rangle \langle 0|$  with  $|0\rangle = \frac{1}{\sqrt{n}} \sum_a |a, \bar{a}\rangle$  being the singlet.

The transfer matrix for the  $SU(n)$  symmetric state described above is given by  $E = \mathcal{P}_{\text{adj}}^\dagger \mathcal{P}_{\text{adj}}$  [35]. Using (29), it is easy to see that  $E = \mathcal{P}_{\text{adj}}$ . The matrix elements of  $E$  read

$$\langle a, \bar{b} | E | c, \bar{d} \rangle = \delta_{ac} \delta^{bd} - \frac{1}{n} \delta_a^b \delta_c^d \quad (30)$$

where the upper indices corresponds to the conjugate representation [36].

As a mapping between vector spaces, the matrix  $E$  can be represented as Fig. (8). From the discussion in the previous subsection, it is clear that this transfer matrix generates operators  $U_i$  satisfying the Temperley-Lieb algebra (22) with  $q = n^2$ .

$$E_{SU(n)} = \text{Diagram of two vertical lines connected by a horizontal line} - \frac{1}{n} \text{Diagram of two blue arcs (one top, one bottom) connected by a horizontal line}$$

FIG. 8. (color online). Graphical representation of transfer matrix  $E$  for  $SU(n)$  VBS. Each blue line represents the action of the  $n \times n$  identity in the corresponding space

Following the arguments of the previous subsection, the Reyni entropy of the AB of  $SU(n)$  VBS maps to a critical  $n^2$ -state Potts model. It is interesting to note that

the  $q$ -state Potts model for  $q > 4$  at criticality undergoes a *first* order phase transition. This means that there is **no** conformal field theory description for the alternating entanglement for this generalized  $SU(n)$  state with  $n > 2$ .

Some insight into this transition can be obtained studying the effective Hamiltonian obtained from the logarithmic derivative of the 2D transfer matrix. This Hamiltonian inherits the symmetries of the transfer matrix  $E$ . In the cases considered here, the effective Hamiltonian should possess  $SU(n)$  symmetry. For  $n = 2$ , the effective Hamiltonian corresponds to the Heisenberg  $XXX$  model. This model is exactly solvable by Bethe ansatz and possess a unique ground state. In the case  $n > 2$ ,  $SU(n)$  antiferromagnetic chains possess a dimerized ground state with a finite gap [37].

The first order transition in the AB entanglement is a signature of this dimerization transition in the effective Hamiltonian, which is essentially the entanglement Hamiltonian. The details of this connection will be published elsewhere [38].

### D. Topological point

The one dimensional transfer matrix (20) at the point  $\beta = 1$  describes a systems whose partition function is not only critical but also topological. The associated classical model becomes a sum over loop configurations in the torus (see Fig (9)). The partition function in this case can be written as

$$Z = \sum_{\mathcal{C}} q^{N(\mathcal{C})/2}. \quad (31)$$

Here the sum is performed over loop configurations, where  $N(\mathcal{C})$  is the total number of loops in the configuration  $\mathcal{C}$ . This point does not correspond to the ground state of the KSZ model for any value of the parameters.

The Hamiltonian whose ground state corresponds to this MPS can be expressed as a local operator acting on spins  $\frac{1}{2}$  ladders [38]. It is important to note that the topological property of the partition function is present for any  $\alpha$ , i.e. any height of the classical model.

Interestingly, in the  $SU(n)$  generalization, the topological point corresponds a group of negative dimension  $n = -1$ . Groups with negative dimension can be properly defined in the context of tensor categories [39].

## III. CONCLUSIONS

In this letter we have analyzed the Reyni entropy of entanglement for an alternating bipartition (AB) of MPS

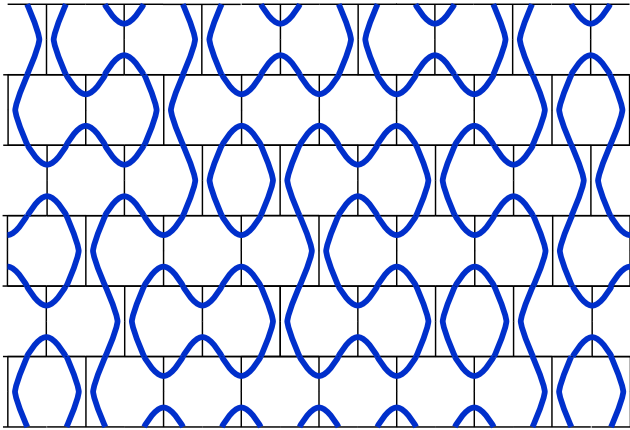


FIG. 9. Loop covering for the Renyi entropy of an alternating bipartition. Here, the number of sites is 8 and  $\alpha = 6$ . Periodic boundary conditions are assumed

states in one dimension. This partition assigns nearest neighbor sites to different complementary sets  $A$  and  $A_c$ . We show the equivalence of the AB Renyi entropy, with the partition function of an eight vertex model for a generic class of MPS states invariant under time reversal. At particular values of the parameters, the MPS possess a  $U(1)$  symmetry and describes the ground state of the KSZ model. In this case the Renyi entropy can be further mapped to a six vertex model. At the fully  $SU(2)$  symmetric point, corresponding to the VBS state, we showed that the classical model describing the Renyi entropy is critical, with central charge  $c = 1$ .

In the  $SU(n)$  generalization of the VBS construction we find that the classical model corresponds to a  $n^2$ -state Potts model at criticality. For  $q > 4$  the  $q$ -state Potts model transition is of first order, signaling in this context a dimerization transition in the entanglement Hamiltonian for the  $SU(n)$  VBS ground state.

We also discuss a special point where the corresponding classical model is not only conformal but also topological. It maps to a loop configuration, which is a quantity that depends on the topology of the lattice. The topological partition function that describes the Reyni ( $\alpha$ ) entropy remains topological even in the limit  $\alpha \rightarrow 1$ , where the Reyni entropy becomes the von-Neumann entropy of entanglement.

Although we have focused in one dimensional systems, the mapping between AB Renyi entropy of a  $D$  dimensional quantum system and a classical partition function in  $D + 1$  dimensions is general.

**Acknowledgments** R. S. is glad to thank the organizers of *Condensed Matter in the City* workshop, where this work was motivated. The author acknowledges P. Coleman, V. Korepin, and T. Wei for useful discussions. The author would also like to thank the anonymous referee for his/her valuable comments. R.S. is supported by grant GIF 1167-165.14/2011.

*Note added:* Upon completion of this work we noticed the results obtained recently in [40, 41] which overlaps with some of the results obtained here. Our result confirms the numerical prediction for the central charge of the associated classical model for the VBS state.

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- [1] X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B **83**, 035107 (2011)
  - [2] A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B **83**, 075102 (2011)
  - [3] L. Fidkowski and A. Kitaev, Phys. Rev. B **83**, 075103 (2011)
  - [4] N. Schuch, D. Pérez-García, and I. Cirac, Phys. Rev. B **84**, 165139 (2011)
  - [5] R. Orús, Annals of Physics **349**, 117 (2014)
  - [6] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Phys. Rev. Lett. **96**, 220601 (2006)
  - [7] T. B. Wahl, H.-H. Tu, N. Schuch, and J. I. Cirac, Phys. Rev. Lett. **111**, 236805 (2013)
  - [8] J. Dubail and R. N., arXiv:1307.7726(2013)
  - [9] F. Verstraete, D. Porras, and J. Cirac, Phys. Rev. Lett. **93**, 227205 (2004)
  - [10] F. Verstraete and J. I. Cirac, Phys. Rev. B **73**, 094423 (2006)
  - [11] T. H. Hsieh and L. Fu, Phys. Rev. Lett. **113**, 106801 (2014)
  - [12] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Phys. Rev. Lett. **59**, 799 (1987)
  - [13] R. J. Baxter, Phys. Rev. Lett. **26**, 832 (1971)
  - [14] H. N. V. Temperley and E. H. Lieb, Proc. R. Soc. A **322**, 251 (1971)
  - [15] F. Haldane, Phys. Lett. A **93**, 464 (1983)
  - [16] U. Schollwöck, Annals of Physics **326**, 96 (2011)
  - [17] D. Perez-Garcia, F. Verstraete, M. M. Wolf, and J. I. Cirac, Quan. Inf. Comp. **7**, 401 (2007)
  - [18] I. Affleck, T. Kennedy, E. H. Lieb, and H. Tasaki, Comm. Math. Phys. **115**, 477 (1988)
  - [19] H. Fan, V. Korepin, and V. Roychowdhury, Phys. Rev. Lett. **93**, 227203 (2004)
  - [20] Y. Xu, H. Katsura, T. Hirano, and V. E. Korepin, J. Stat. Phys. **133**, 347 (2008)
  - [21] H. Katsura, N. Kawashima, A. N. Kirillov, V. E. Korepin, and S. Tanaka, J. Phys. A. **43**, 255303 (2010)
  - [22] V. E. Korepin and Y. Xu, IJMPB **24**, 1361 (2010)
  - [23] L. Amico, R. Fazio, A. Osterloh, and V. Vedral, Rev. Mod. Phys. **80**, 517 (2008)
  - [24] R. A. Santos, F. N. C. Paraan, V. E. Korepin, and A. Klümper, Europhys. Lett. **98**, 37005 (2012)
  - [25] M. B. Hastings, J. Stat. Mech. **2007**, P08024 (2007)
  - [26] J. Eisert, M. Cramer, and M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010)
  - [27] A. Klümper, A. Schadschneider, and J. Zittartz, Z. Phys. B **87**, 281 (1992)
  - [28] A. Klümper, A. Schadschneider, and J. Zittartz, Europhys. Lett. **24**, 293 (1993)
  - [29] E. H. Lieb, Phys. Rev. **162**, 162 (1967)

- [30] R. Baxter., *Exactly solved models in statistical mechanics* (Academic Press, London - New York, 1982)
- [31] A. Belavin, A. Polyakov, and A. Zamolodchikov, Nucl. Phys. B **241**, 333 (1984)
- [32] H. W. J. Blöte, J. L. Cardy, and M. P. Nightingale, Phys. Rev. Lett. **56**, 742 (1986)
- [33] M. Fujimoto, J. Stat. Phys. **82**, 1519 (1996)
- [34] H. Katsura, T. Hirano, and V. E. Korepin, Journal of Physics A: Mathematical and Theoretical **41**, 135304 (2008)
- [35] R. Orús and H.-H. Tu, Phys. Rev. B **83**, 201101 (2011)
- [36] In 20, we do not distinguish between upper and lower indices, as the fundamental representation and its conjugate are equivalent for  $SU(2)$
- [37] I. Affleck, Journal of Physics: Condensed Matter **2**, 405 (1990)
- [38] R. A. Santos *to be published*
- [39] R. Penrose, in *Combinatorial Mathematics and its applications*, edited by D. Welsh (Academic Press, New York, 1971)
- [40] W.-J. Rao, X. Wan, and G.-M. Zhang, Phys. Rev. B **90**, 075151 (2014)
- [41] T. H. Hsieh, L. Fu, and X.-L. Qi, Phys. Rev. B **90**, 085137 (2014)